♦ **Exercice 1. Homotopy Invariance.** The objective is to (re)prove that singular homology is homotopy invariant.

1. Let $F, G\colon \mathcal{C} \to Ch_\varepsilon(R)$ be two functors that are acyclic and free relative to the models $\mathcal{M}$. Show that the natural transformations $\tau\colon F \to G$ and $\sigma\colon G \to F$ provided by the acyclic models Theorem consist of homotopy equivalences τ_X and σ_X for any $X \in \mathcal{C}$.
2. Let X be a non-empty space. We write $i_k(X)$ for the inclusions $X \to X \times I$ given by $x \mapsto (x, k)$ with $k = 0, 1$. Show that there exists a chain homotopy $h(X)$ between $i_0(X)_\bullet$ and $i_1(X)_\bullet$ the maps induced by the inclusions on singular chain complexes.
3. With the same notation as above, prove that, for any map $f\colon X \to Y$ the two morphisms $(f \times Id_I)_\bullet \circ h(X)$ and $h(Y) \circ f_\bullet$ are equal.
4. Prove that two homotopic maps $f, g\colon X \to Y$ induce the same map in homology $H_*(-; R)$.
5. Prove the relative version of this statement (depending on the time, give only a skecth of the proof).

♦ **Exercice 2. Induced and coinduced modules.** Let G be a group and A any abelian group. Recall that the induced module $\operatorname{Ind} A$ is $\mathbb{Z}G \otimes_{\mathbb{Z}} A$ and the coinduced module $\operatorname{CoInd} A = \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}G, A)$ with G acting on the left on $\mathbb{Z}G$.

1. Show that $H^n(G; -)$ vanishes on coinduced modules and that $H_n(G; -)$ vanishes on induced modules for $n \geq 1$.
2. Let $\partial_{g,a}\colon \mathbb{Z}G \to A$ denote the homomorphism that sends g to a and any other group element to zero. Construct a G-equivariant homomorphism $\operatorname{Ind} A \to \operatorname{CoInd} A$ and prove it is injective.
3. Let G be a finite group. Show that the map constructed in 2. is surjective.
4. Let G be a finite group. Conclude that $H^n(G; -)$ vanishes on induced modules for $n \geq 1$.

Remark. This second exercise complements and maybe clarifies Exercise 2 on Sheet 7.

♦ indicates the weekly assignments, designed for a 20-25 minute long presentation by a group of two.